HW 1 P13

3. Since $Re(z_1+z_2) \le |Re(z_1+z_2)|$

$$\le \sqrt{|Re(z_1+z_2)|^2 + |Im(z_1+z_2)|^2}$$

$$= |z_1+z_2|$$

$$\le |z_1|+|z_2|$$

By Reverse triangle inequality,

$$|z_3+z_4| = |z_3 - (-z_4)|$$

$$\geq \big||z_3| - |z_4|\big|$$

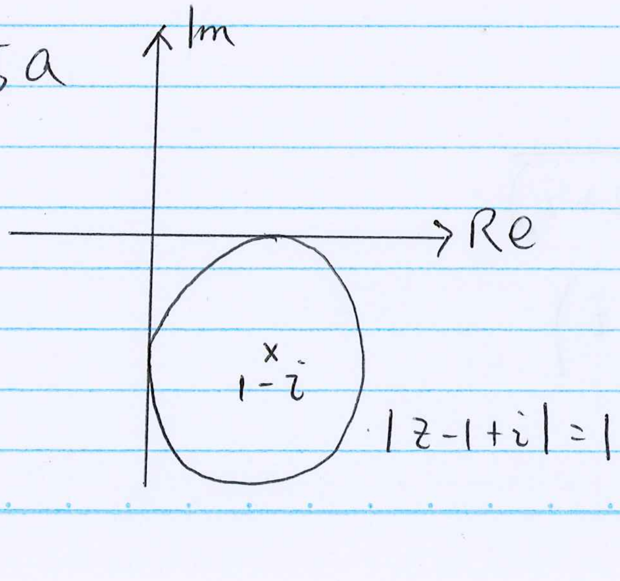
Thus, $\dfrac{Re(z_1+z_2)}{|z_3+z_4|} \le \dfrac{|z_1|+|z_2|}{\big||z_3|-|z_4|\big|}$

4 let $z = x+yi$, then $(\sqrt{2}|z|)^2 = 2|z|^2 = 2x^2+2y^2$

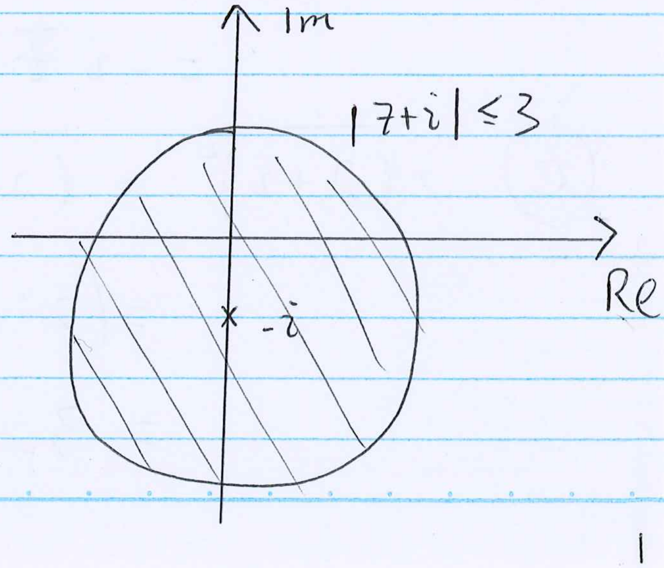
$$(\sqrt{2}|z|)^2 - \big(|Re(z)| + |Im(z)|\big)^2 = x^2+y^2-2xy$$

$$= (x-y)^2 \geq 0.$$

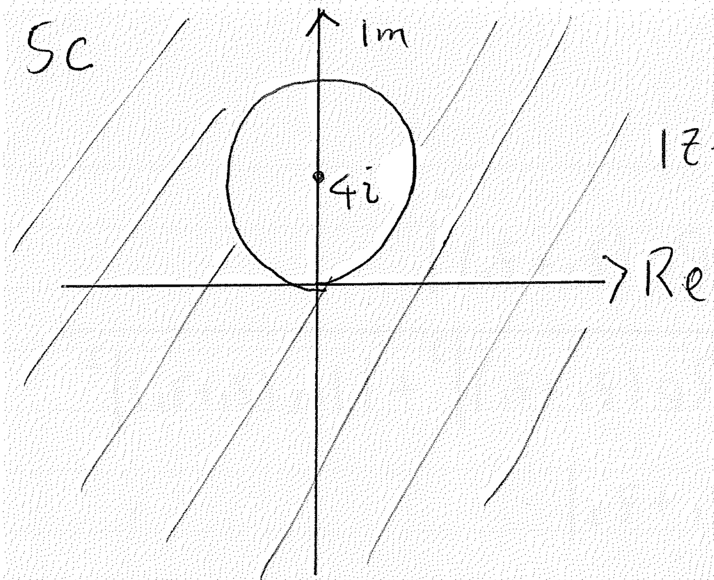
5a



5b



5c



$|z-4i| \geq 4.$

6. Since $z=0$ satisfies the equation $|z-1| = |z-i|$. For $z=-1+i$,

$$|-1+i-1| = |-2+i| = \sqrt{5}$$

$$|-1+i+i| = |-1+2i| = \sqrt{5}$$

$z=-1+i$ also lies in the line, so results follows.

P16

① a $\quad \overline{z+3i} = \bar{\bar{z}} + \overline{(+3i)}$

$$= z - 3i$$

ⓑ $\quad \overline{iz} = (\bar{i})(\bar{z})$

$$= -i\,\bar{z}$$

ⓒ $\quad \overline{(2+i)^2} = \overline{(2+i)}\;\overline{(2+i)}$

$$= (2-i)(2-i)$$

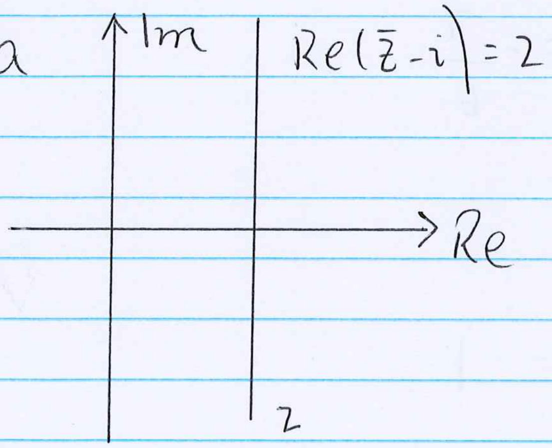
$$= 3-4i$$

d $\quad |(2\bar{z}+5)(\sqrt{2}-i)| = |\overline{(2\bar{z}+5)(\sqrt{2}-i)}|$

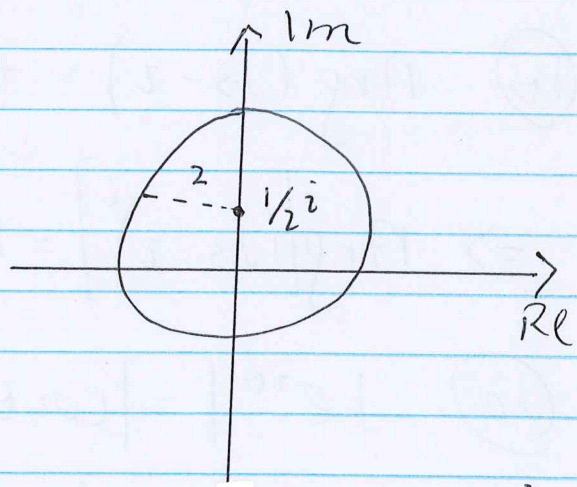
$$= |(2z+5)(\sqrt{2}+i)|$$

$$= |(2z+5)|\sqrt{3}$$

2 a



b



Circle centre at $\frac{1}{2}i$ with radius 2.

7 $\quad |Re(2+\bar{z}+z^3)| \leq |2+\bar{z}+z^3|$

$$\leq |2| + |\bar{z}| + |z^3| \leq 4.$$

9 $\quad \left|\dfrac{1}{z^4-4z^2+3}\right| = \dfrac{1}{|z^2-3||z^2-1|} \leq \dfrac{1}{||z|^2-3|||z|^2-1|}$

$$= \frac{1}{3} \text{ if } |z|=2$$

P23-24

(1a) $z = -\frac{2}{1+\sqrt{3}i} = \frac{-2(1-\sqrt{3}i)}{2} = -1+\sqrt{3}i$

$\mathrm{Arg}(z) = \tan^{-1}(-\sqrt{3}) = 2\pi/3$.

(1b) $\mathrm{Arg}(\sqrt{3}-i) = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\frac{\pi}{6}$

$\Rightarrow \mathrm{Arg}\left[(\sqrt{3}-i)^6\right] = \pi$.

(2a) $|e^{i\theta}| = |\cos\theta + i\sin\theta| = 1$

(2b) $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$

$= \cos(-\theta) + i\sin(-\theta)$

$= e^{-i\theta}$

(4) $|e^{i\theta}-1|^2 = 4$

$(\cos\theta - 1)^2 + \sin^2\theta = 4$

$1 - 2\cos\theta + 1 = 4$

$\cos\theta = -1$

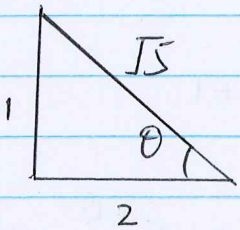
$\theta = \pi$.

(5a) $i = e^{\pi/2\, i}$, $(1-\sqrt{3}i) = 2e^{-\pi/3\, i}$

$\sqrt{3}+i = 2e^{\pi/6\, i}$

$\Rightarrow i(1-\sqrt{3}i)(\sqrt{3}+i) = 4e^{\pi/3\, i} = 2(1+\sqrt{3}i)$.

5b L.H.S. : $i = e^{\pi/2\, i}$, $2+i = \sqrt{5}\, e^{\theta i}$ with $\theta = \tan^{-1} 1/2 \in (0, \pi/2)$



$$\frac{5i}{2+i} = \sqrt{5}\, e^{(\pi/2 - \theta) i}$$

$$= \sqrt{5}\, e^{(\pi/2 - \tan^{-1} 1/2) i}$$

$$= \sqrt{5} \left(\cos\left(\pi/2 - \tan^{-1} 1/2\right) - i \sin\left(\pi/2 - \tan^{-1} 1/2\right) \right)$$

$$= 1 + 2i$$

5c $\sqrt{3} + i = 2e^{\pi/6\, i} \Rightarrow (\sqrt{3}+i)^6 = -64$.

5d $(1+\sqrt{3} i) = 2e^{\pi/3\, i} \Rightarrow (1+\sqrt{3} i)^{-10} = 2^{-10} e^{-10\pi/3\, i}$

$$= 2^{-10} e^{2\pi/3\, i}$$

$$= 2^{-11}(-1+\sqrt{3} i).$$

7 $z^m = r^m e^{im\theta}$

$$(z^m)^{-1} = r^{-m} e^{i(-m\theta)}$$

$$= r^{-m} e^{im(-\theta)}$$

$$= (r^{-1} e^{-i\theta})^m$$

$$= (z^{-1})^m$$

(9) $(1+z+z^2+\dots+z^n)(1-z)$

$= (1+z+z^2+\dots+z^n) - (z+z^2+z^3+\dots+z^{n+1})$

$= 1-z^{n+1}$.

let $z = e^{i\theta} = \cos\theta + i\sin\theta$

L.H.S. $= 1 + e^{i\theta} + e^{2i\theta} + \dots + e^{2in\theta}$

Real part of L.H.S. $= 1+\cos\theta + \cos 2\theta + \dots + \cos n\theta$

R.H.S. $= \dfrac{1-z^{n+1}}{1-z}$

$$= \frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}}$$

$$= \frac{1-\cos(n+1)\theta - i\sin(n+1)\theta}{(1-\cos\theta - i\sin\theta)} \cdot \frac{1-\cos\theta+i\sin\theta}{1-\cos\theta+i\sin\theta}$$

Thus real part of R.H.S. equals to.

$$\frac{(1-\cos(n+1)\theta)(1-\cos\theta) + \sin(n+1)\theta\sin\theta}{(1-\cos\theta)^2+\sin^2\theta}$$

$$= \frac{1-\cos\theta - \cos(n+1)\theta + \cos\theta\cos(n+1)\theta + \sin(n+1)\theta\sin\theta}{2-2\cos\theta}$$

$$= \frac{1}{2} + \frac{-\cos(n+1)\theta + \cos n\theta}{2(1-\cos\theta)} = \frac{1}{2} + \frac{\sin(2n+1)\theta/2}{2\sin\theta/2}$$

P30-31

(1a) let square root of $2i$ be z,

$$z^2 = 2i = 2e^{\pi/2\, i}$$

$$z = \sqrt{2}\, e^{(\pi/2 + 2n\pi)\frac{1}{2} i} \quad \text{for } n=0,1$$

$$= \sqrt{2}\, e^{\frac{\pi}{4} i},\ \sqrt{2}\, e^{5\pi/4\, i}$$

$$= \pm(1+i)$$

(1b) let the square root of $1-\sqrt{3}i$ be z

$$z^2 = 1-\sqrt{3}i = 2e^{-\pi/3\, i}$$

$$z = \sqrt{2}\, e^{(-\pi/3 + 2n\pi)\frac{1}{2} i} \quad \text{for } n=0,1$$

$$= \sqrt{2}\, e^{\frac{-\pi}{6} i},\ \sqrt{2}\, e^{\frac{5\pi}{6} i}$$

$$= \pm \frac{\sqrt{3} - i}{\sqrt{2}}$$

(2)

$$z^3 = -8i = 8e^{\frac{3\pi}{2} i}$$

$$z = 2e^{(3\pi/2 + 2n\pi)\frac{1}{3} i} \quad n=0,1,2$$

$$= 2e^{\pi/2\, i},\ 2e^{\frac{7\pi}{6} i},\ 2e^{\frac{11\pi}{6} i}$$

$$= 2i,\ \pm\sqrt{3} - i.$$

③ $z^4 = -8 - 8\sqrt{3}i$

$= 16e^{-2\pi/3\, i}$

$z = 2e^{(-2\pi/3 + 2\pi n)\frac{1}{4}i} \qquad n = 0, 1, 2, 3$

$= 2e^{\frac{-\pi}{6}i}, 2e^{i\pi/3}, 2e^{5\pi/6\, i}, 2e^{-2\pi/3\, i}$

$= \pm(\sqrt{3} - i), \pm(1 + \sqrt{3}i).$

④ a $z^3 = -1 = e^{\pi i}$

$z = e^{(\pi + 2n\pi)\frac{1}{3}i} \qquad n = 0, 1, 2$

$= e^{\frac{\pi}{3}i}, e^{\pi i}, e^{5\pi/3\, i}$

b $z^6 = 8 = 8e^0$

$z = 8^{1/6} e^{\frac{n\pi}{3}i} \qquad n = 0, 1, 2 \dots 5$

$= \pm\sqrt{2}, \pm\frac{1 + \sqrt{3}i}{\sqrt{2}}, \pm\frac{1 - \sqrt{3}i}{\sqrt{2}}.$

⑥ $z^4 = -4 = 4e^{i\pi}$

$z = \sqrt{2}e^{(\pi + 2n\pi)\frac{1}{4}i} \qquad n = 0, 1, 2, 3$

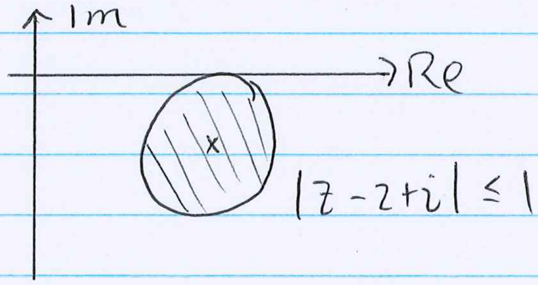
$= \sqrt{2}e^{i\pi/4}, \sqrt{2}e^{3\pi/4\, i}, \sqrt{2}e^{5\pi/4\, i}, \sqrt{2}e^{7\pi/4\, i}$

Thus, $z^4 + 4 = \left[(z - \sqrt{2}e^{i\pi/4})(z - \sqrt{2}e^{i3\pi/4})\right]\left[(z - \sqrt{2}e^{\frac{5\pi}{4}i}) \cdot (z - \sqrt{2}e^{7\pi/4\, i})\right]$

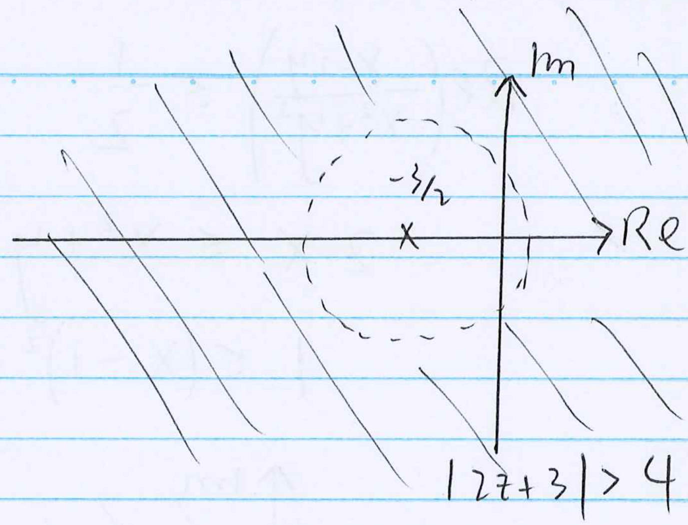
$= (z^2 + 2z + 2)(z^2 - 2z + 2).$

P34-35

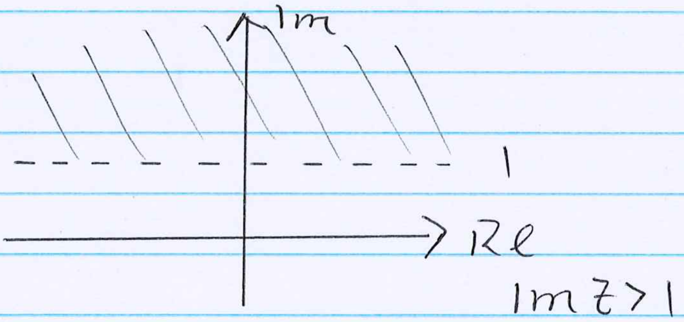
1a



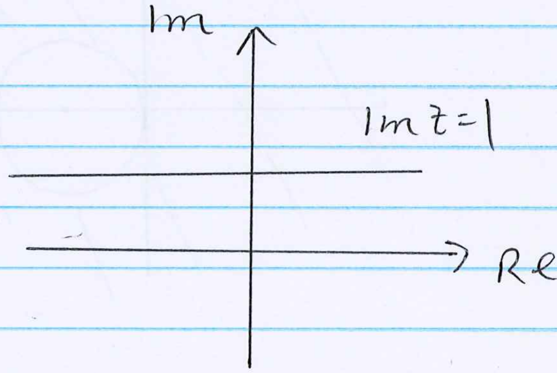
1b



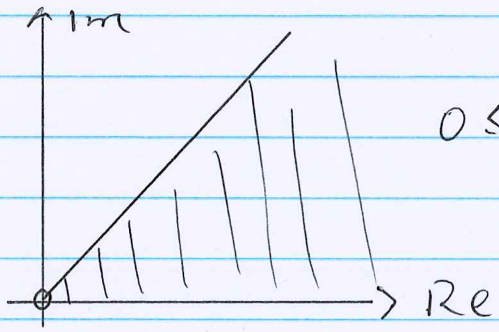
1c



1d



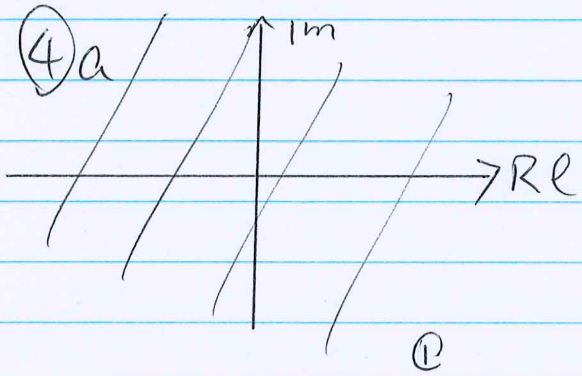
1e



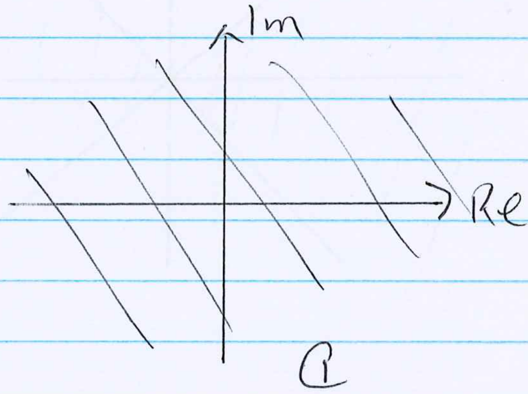
$0 \le \arg z \le \pi/4$

b, c are domains.

4a



4b

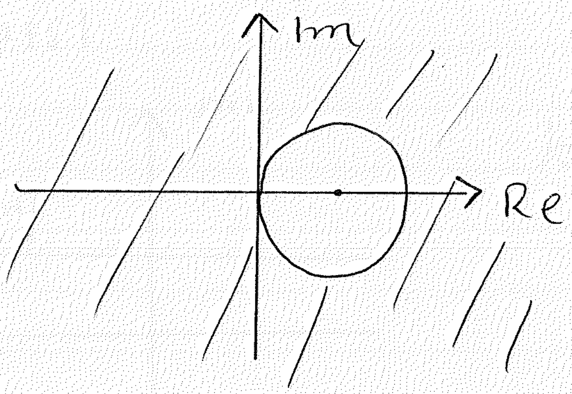


4c $Re\left(\frac{1}{z}\right) \leq \frac{1}{2}$

$$Re\left(\frac{x-iy}{x^2+y^2}\right) \leq \frac{1}{2}$$

$$2x \leq x^2+y^2$$

$$1 \leq (x-1)^2+y^2$$



4d $z^2 = x^2-y^2+2xyi$

$$Re(z^2) = x^2-y^2 > 0$$